1 Introduction

There are, to the best of our knowledge, four forces at play in the Universe. At the very largest scales - those of planets or stars or galaxies - the force of gravity dominates. At the very smallest distances, the two nuclear forces, strong and week, dominate. For everything in between, it is force of electromagnetism that rules.

At the atomic scale, electromagnetism with some quantum effects governs the interactions between atoms and molecules. It is the force that underlies the periodic table of elements, giving rise to all of chemistry and, through this, much of biology. It is the force which binds atoms together into solids and liquids. And it is the force which is responsible for the incredible range of properties that different materials manifest.

At the macroscopic scale, electromagnetism manifests itself in the familiar phenomena that give the force its name. In the case of electricity, this means everything from rubbing a balloon on your head and sticking it on the wall, through to the fact that you can plug any appliance into the wall and be pretty confident that it will work. For magnetism, this means everything from the ATM card to the computer hard disk ... to the trains in Japan which runs above the rail (do a google search on Maglev Train). Harnessing these powers through the invention of the electric dynamo and motor has transformed the planet and our lives on it.

As if this wasn't enough, there is much more to the force of electromagnetism for it is, quite literally, responsible for everything you've ever seen. It is the force that gives rise to light itself.

Rather remarkably, a full description of the force of electromagnetism is contained in four simple and elegant equations. These are known as the Maxwell equations. There are few places in physics, or indeed in any other subject, where such a richly diverse set of phenomena flows from so little. Understanding the mathematical beauty of the equations will allow us to see some of the principles that underly the laws of physics, laying the groundwork for future study of the other forces.

1.1 Charge and Current

Each particle in the Universe carries with it a number of properties. These determine how the particle interacts with each of the four forces. For the force of gravity, this property is mass. For the force of electromagnetism, the property is called electric charge.

Importantly, charge can be positive or negative. It can also be zero, in which case the particle is unaffected by the force of electromagnetism. The SI unit of charge is the Coulomb, denoted by C. Charge is quantised: the charge of any particle is an integer multiple of the charge carried by the electron which we denoted as $-e$, with

$$e = 1.60217657 \times 10^{-19}\ C$$

We need to move beyond the dynamics of point particles and onto the dynamics of continuous objects known as fields. To aid in this, it's useful to consider the charge density,

$$\rho(\mathbf{x}, t)$$

defined as charge per unit volume. The total charge Q in a given region V is simply $Q = \int_V d^3x \rho(\mathbf{x}, t)$. In most situations, we will consider smooth charge densities, which can be thought of as arising from averaging over many point-like particles. But, on occasion, we will return to the idea of a single particle of charge q, moving on some trajectory $\mathbf{r}(t)$, by writing $\rho = q\delta(\mathbf{x} - \mathbf{r}(t))$ where the delta-function ensures that all the charge sits at a point.

More generally, we will need to describe the movement of charge from one place to another. This is captured by a quantity known as the current density $\mathbf{J}(\mathbf{x}, t)$, defined as follows: for every surface S, the integral

$$I = \int_S \mathbf{J} \cdot d\mathbf{S}$$

counts the charge per unit time passing through S. (Here $d\mathbf{S}$ is the unit normal to S). The quantity I is called the current. In this sense, the current density is the current-per-unit-area.

The above is a rather indirect definition of the cur- rent density. To get a more intuitive picture, consider a continuous charge distribution in which the velocity of a small volume, at point $\mathbf{x}$, is given by $\mathbf{v}(\mathbf{x}, t)$. Then, neglecting relativistic effects, the current density is

$$\mathbf{J} = \rho\mathbf{v}$$

In particular, if a single particle is moving with velocity $\mathbf{v} = \dot{\mathbf{r}}(t)$, the current density will be $\mathbf{J} = q\mathbf{v}\delta^3(\mathbf{x} - \mathbf{r}(t))$

This is illustrated in the figure, where the underlying charged particles are shown as red balls, moving through the blue surface S.

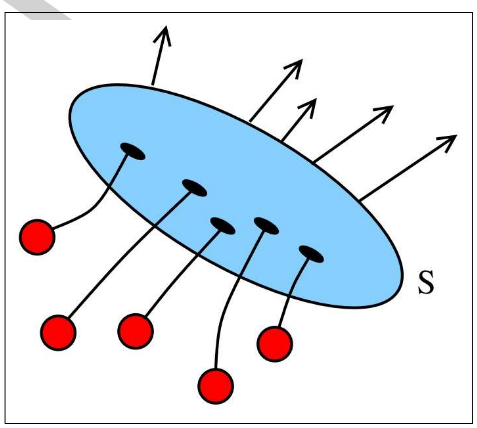


Figure 1.1: Current Flux

As a simple example, consider electrons moving along a wire. We model the wire as a long cylinder of cross-sectional area A as shown below. The electrons move with velocity $\mathbf{v}$, parallel to the axis of the wire. (In reality, the electrons will have some distribution of speeds; we take $\mathbf{v}$ to be their average velocity). If there are n electrons per unit volume, each with charge q, then the charge density is $\rho = nq$ and the current density is $\mathbf{J} = nq\mathbf{v}$. The current itself is $I = |\mathbf{J}|A$.

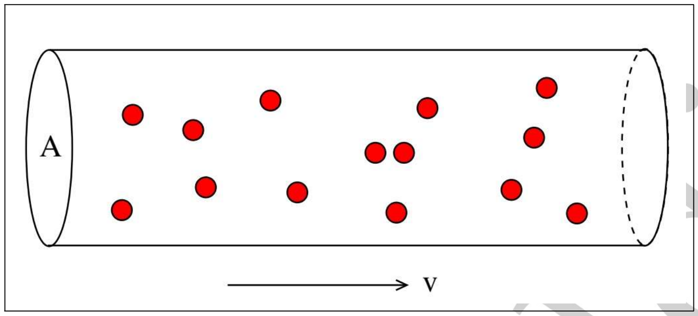


Figure 1.2: Current Wire

1.2 Conservation Law

The most important property of electric charge is that it's conserved. This, of course, means that the total charge in a system can't change. But it means much more than that because electric charge is conserved locally. An electric charge can't just vanish from one part of the Universe and turn up somewhere else. It can only leave one point in space by moving to a neighbouring point.

The property of local conservation means that ρ can change in time only if there is a compensating current flowing into or out of that region. We express this in the continuity equation,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0 \tag{1.1}$$

This is an important equation. It arises in any situation where there is some quantity that is locally conserved.

To see why the continuity equation captures the right physics, it's best to consider the change in the total charge Q contained in some region V.

$$\frac{dQ}{dt} = \int_V d^3x \frac{\partial \rho}{\partial t} = -\int_V d^3x \nabla \cdot \mathbf{J} = -\int_S \mathbf{J} \cdot d\mathbf{S}$$

From our previous discussion, $\int_S \mathbf{J} \cdot d\mathbf{S}$ is the total current flowing out through the boundary S of the region V. (It is the total charge flowing out, rather than in, because $d\mathbf{S}$ is the outward normal to the region V). The minus sign is there to ensure that if the net flow of current is outwards, then the total charge decreases.

If there is no current flowing out of the region, then $dQ/dt = 0$. This is the statement of (global) conservation of charge. In many applications we will take V to be all of space, $\mathbf{R}^3$, with both charges and currents localised in some compact region. This ensures that the total charge remains constant.

1.3 Force and Fields

Any particle that carries electric charge experiences the force of electromagnetism. But the force does not act directly between particles. Instead, Nature chose to introduce intermediaries. These are fields.

In physics, a "field" is a dynamical quantity which takes a value at every point in space and time. To describe the force of electromagnetism, we need to introduce two fields, each of which is a three-dimensional vector. They are called the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$,

$$\mathbf{E}(\mathbf{x}, t) \quad \text{and} \quad \mathbf{B}(\mathbf{x}, t)$$

When we talk about a "force" in modern physics, we really mean an intricate interplay between particles and fields. There are two aspects to this. First, the charged particles create both electric and magnetic fields. Second, the electric and magnetic fields guide the charged particles, telling them how to move. This motion, in turn, changes the fields that the particles create. We're left with a beautiful dance with the particles and fields as two partners, each dictating the moves of the other.

This dance between particles and fields provides a paradigm which all other forces in Nature follow. It feels like there should be a deep reason that Nature chose to introduce fields, associated to all the forces. And, indeed, this approach does provide one overriding advantage: all interactions are local. Any object, whether particle or field affects things only in its immediate neighbourhood. This influence can then propagate through the field to reach another point in space, but it does not do so instantaneously. It takes time for a particle in one part of space to influence a particle elsewhere. This lack of instantaneous interaction allows us to introduce forces which are compatible with the theory of special relativity, something that we will explore in more detail later.

The position $\mathbf{r}(t)$ of a particle of charge q is dictated by the electric and magnetic fields through the Lorentz force law,

$$\mathbf{F} = q(\mathbf{E} + \dot{\mathbf{r}} \times \mathbf{B})$$

The motion of the particle can then be determined through Newton's equation $\mathbf{F} = m\ddot{\mathbf{r}}$. Roughly speaking, an electric field accelerates a particle in the direction $\mathbf{E}$, while a magnetic field causes a particle to move in circles in the plane perpendicular to $\mathbf{B}$.

We can also write the Lorentz force law in terms of the charge distribution $\rho(\mathbf{x}, t)$ and the current density $\mathbf{J}(\mathbf{x}, t)$. Now we talk in terms of the force density $\mathbf{f}(\mathbf{x}, t)$, which is the force acting on a small volume at point $\mathbf{x}$. Now the Lorentz force law reads

$$\mathbf{f} = \rho\mathbf{E} + \mathbf{J} \times \mathbf{B} \tag{1.2}$$

1.4 Maxwell's equation

In this course, most of our attention will focus on the other side of the dance: the way in which electric and magnetic fields are created by charged particles. This is described by a set of four equations, known collectively as the Maxwell equations. They are:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{1.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{1.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{1.5}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \tag{1.6}$$

The equations involve two constants. The first is the electric constant (known also, in slightly old-fashioned terminology, as the permittivity of free space),

$$\epsilon_0 \approx 8.85 \times 10^{-12} m^{-3} Kg^{-1} s^2 C^2$$
$$\frac{1}{4\pi\epsilon_0} \approx = 9 \times 10^9$$

It can be thought of as characterising the strength of the electric interactions. The other is the magnetic constant (or permeability of free space),

$$\mu_0 = 4\pi \times 10^{-7} mKgC^{-2}$$
$$\approx 1.25 \times 10^{-6} mKgC^{-2}$$

We will slowly understand the importance of these equations in depth as the lectures proceed. Before we start our journey let us discuss which book we need to read.

1.5 Books required

Most of you have read, at least party, David Griffith's Electrodynamics book. This is remains to be the core text of our lectures. In CSIR NET in general 5 to 6 questions are asked in the exam. Most of the problems in NET, GATE and JEST exams from Electrodynamics part can be answered if you have studied Griffiths properly and done the exercises of the book .

You need to have sound understanding of vectors, differential equations, some of the special functions, dirac delta functions to understand the mathematics of the ED. I recommend M.L Boas and Riley Hobson Bence's Mathematical Methods book for these parts.

2 Electrostatics

In this section, we will be interested in electric charges at rest. This means that there exists a frame of reference in which there are no currents; only stationary charges. Of course, there will be forces between these charges but we will assume that the charges are pinned in place and cannot move. The question that we want to answer is: what is the electric field generated by these charges?

2.1 Coulomb's Law

The basic principle of electrostatics is based on the fact the electric charges attract or repel other charges depending on their relative signs and the law of force is given by Coulomb's law.

The form of the law does not depend on the choice of origin. The force on the charge q_2 located at the position $\vec{r_2}$ due to a charge q_1 located at the position $\vec{r_1}$ is proportional to the product of the charges q_1 and q_2 and is inversely proportional to the square of the distance r between the charges. In vector form, the form of Newton's law is written as

$$\overrightarrow{F_{12}} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\hat{r}}{r^2} \tag{2.1}$$

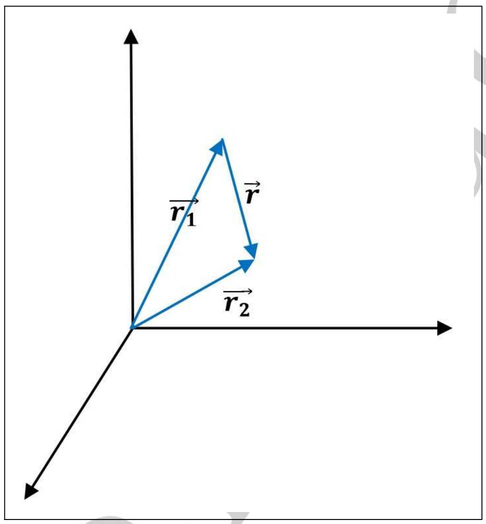


Figure 2.1: Coulomb's law and direction

We may note a few things about Coulomb's law. If $\overrightarrow{F_{12}}$ is the force on charge q_2 due to the charge q_1, by Newton's third law, the force on the charge q_1 due to q_2 is equal and opposite $\overrightarrow{F_{21}} = \overrightarrow{F_{12}}$.

There is an another way to write the law, as you see in the picture. Let $\mathbf{r}_1$ and $\mathbf{r}_2$ denote the positions of q_1 and q_2 with respect to some origin in space. Then the relative vector $\mathbf{r}$ from q_2 to q_1 is $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$, and the force between q_1 and q_2 is

$$\mathbf{F}_{12} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{r}_1 - \mathbf{r}_2}{|\mathbf{r}_1 - \mathbf{r}_2|^3} \tag{2.2}$$

In most of the times it will also be written as

$$\mathbf{F} = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\mathbf{x}_1 - \mathbf{x}_2}{|\mathbf{x}_1 - \mathbf{x}_2|^3} \tag{2.3}$$

2.2 Electric Field

The definition of $\mathbf{E}(\mathbf{x})$ is the force per unit charge that would be exerted on a small "test charge" q if it were located at $\mathbf{x}$, in the limit $q \to 0$

$$\mathbf{E}(\mathbf{x}) = \lim_{q \to 0} \frac{\mathbf{F}}{q} \tag{2.4}$$

The test charge is taken to be small so that it does not affect the other charges. Because the force F on q is proportional to q, the electric field is independent of the test charge. This definition provides a technique in principle for measuring the defined quantity $\mathbf{E}(\mathbf{x})$: Take a small charge to $\mathbf{x}$ and measure $\mathbf{F}/q$

The electric field exerts a force on a charged particle q, given by

$$\mathbf{F}_q = q\mathbf{E}(\mathbf{x}) \tag{2.5}$$

2.2.1 Superposition principle

If the source of the electric field is due to multiple charges, the field is simply the vector sum of the fields due to each field. This is known as the superposition principle.

In the case of multiple charges, q_i positioned at $\vec{r}_i$, the total electric field just add up vectorically.

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{N} \frac{q_i\left(\vec{r} - \vec{r}_i\right)}{\left|\vec{r} - \vec{r}_i\right|^3} \tag{2.6}$$

For continuous charge distribution

$$\mathbf{E}(\mathbf{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\mathbf{x} - \mathbf{x}'}{\left|\mathbf{x} - \mathbf{x}'\right|^3} \rho\left(\mathbf{x}'\right) d^3x' \tag{2.7}$$

Example: Figure 2.2 below shows six identical charges, one at each vertex of a regular hexagon in the xy plane centered at the origin.

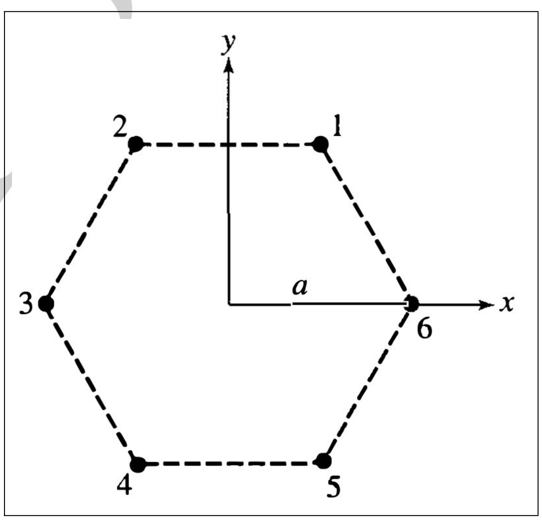


Figure 2.2: Six charges at the corner of a regular hexagon

(a) What is the electric field at any field point x in the xy plane? (b) What is the field on the x axis? (c) As an example of asymptotic approximation, what is the field on the x axis far from the origin, accurate through order x^{-4}? (d) If the 6 th charge is removed from the group, what is the field at the origin for the remaining 5 charges ($k = 1, 2, \ldots, 5$)?

Solution: (a) Let the charges be numbered with the index k from 1 to 6. Then the position of the k th charge is

$$\mathbf{x}'_k = \hat{\mathbf{i}} a \cos\frac{k\pi}{3} + \hat{\mathbf{j}} a \sin\frac{k\pi}{3}$$

The field for arbitrary $\mathbf{x}$ is then obtained from superposition prrinciple by adding the contribution from all the charges

$$\mathbf{E}(\mathbf{x}) = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{6}\frac{(x - a\cos k\pi/3)\hat{\mathbf{i}} + (y - a\sin k\pi/3)\hat{\mathbf{j}}}{\left[(x - a\cos k\pi/3)^2 + (y - a\sin k\pi/3)^2\right]^{3/2}} \tag{2.8}$$

where a is the distance from the origin to any of the charges. The field is singular, i.e., goes to infinity, at the six charges, but is finite everywhere else. (b) We set $y = 0$ in equation (2.8) and get

$$E_x(x,0) = \frac{q}{4\pi\epsilon_0}\sum_{k=1}^{6}\frac{(x - a\cos k\pi/3)}{\left[x^2 - 2ax\cos k\pi/3 + a^2\right]^{3/2}} \tag{2.9}$$

$$E_y(x,0) = \frac{-qa}{4\pi\epsilon_0}\sum_{k=1}^{6}\frac{\sin k\pi/3}{\left[x^2 - 2ax\cos k\pi/3 + a^2\right]^{3/2}} = 0 \tag{2.10}$$

It is clear that $E_y(x,0)$ must be zero, because of the symmetric positions of the charges; in the sum in the last equation (2.10) terms with $k = 3$ and 6 are 0 and the other terms cancel in pairs.

At the origin, $\mathbf{E}$ is 0, which is easily seen from the equations or just physically because the fields due to charges on opposite vertices cancel.

(c) The y component is zero on the x axis. For $x \gg a$, the x component must be calculated by expanding (2.9) in a power series in the small quantity a/x using the method of Taylor series. The result is

$$E_x(x,0) \approx \frac{1}{4\pi\epsilon_0}\left\{\frac{6q}{x^2} + \frac{9qa^2}{2x^4}\right\} \tag{2.11}$$

The leading term is, as expected, the same as for a point charge $6q$ at the origin.

(d) We could start from (2.9) but sum only from $k = 1$ to 5. However, the superposition principle enables us to answer the question more easily if we realize that the field of charges 1 to 5 is the same as the field of charges 1 to 6 superposed on the field of a charge $-q$ at $(a, 0)$. But the field at the origin due to the original 6 charges is 0, so the field of charges 1 to 5 is (note the sign!)

$$\mathbf{E}_{\text{5charges}}(0,0) = \frac{+q\hat{\mathbf{i}}}{4\pi\epsilon_0 a^2} \tag{2.12}$$

Example: What is the electric field on the midplane of a uniformly charged thin wire of length 2ℓ? The charge per unit length is λ.

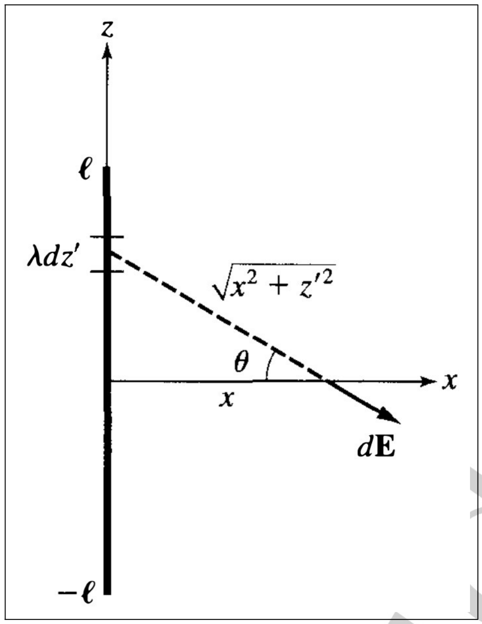


Figure 2.3: A charged wire. $(0,0,z')$ is a source point and $(x,0,0)$ is a field point.

Figure 2.3 shows the wire extending from $(0,0,-\ell)$ to $(0,0,+\ell)$ along the z axis. Because of axial symmetry it is sufficient to find the electric field on the x axis. As shown, the charge element $\lambda dz'$ produces field $d\mathbf{E}$ at the point $(x,0,0)$ Because the distribution is symmetric about $z=0$, the resultant field due to the entire wire will be in the x direction. The x component of $\mathbf{E}$ is, by integrating

$$E_x(x,0,0)=\frac{1}{4\pi\epsilon_0}\int_{-\ell}^{\ell}\frac{x\lambda dz'}{\left(x^2+z'^2\right)^{3/2}}=\frac{\lambda\ell}{2\pi\epsilon_0 x\sqrt{x^2+\ell^2}} \tag{2.13}$$

We might derive this result in another, more geometrical way. The contribution dE_x due to $dq'=\lambda dz'$ is $\hat{\mathbf{i}}\cdot d\mathbf{E}=dE\cos\theta$, where $\cos\theta=x/\sqrt{x^2+z^2}$. Then E_x is $\int\cos\theta dE$, leading again to the same result.

Note that the field in the midplane is radial. Generalizing to any point on the xy plane, at distance r from the z axis,

$$\mathbf{E}(r)=\frac{\lambda\ell\hat{\mathbf{r}}}{2\pi\epsilon_0 r\sqrt{r^2+\ell^2}} \tag{2.14}$$

where $\hat{\mathbf{r}}$ is the radial unit vector in the xy plane.

It is interesting to consider the limiting behavior of this field for far points and near points. Far from the line charge, i.e., $r\gg\ell$, the field is $E_r=\lambda\ell/\left(2\pi\epsilon_0 r^2\right)$

As we'd expect, this is the same as the field of a point charge $q=2\ell\lambda$ at the origin; from far away the line charge looks, to a first approximation, like a point. Near the wire, i.e., $r\ll\ell$, the field is $E_r=\lambda/\left(2\pi\epsilon_0 r\right)$.

If the line charge is infinititly long, then the field on the x axis is obtained from equation (2.13) by extending the integral from $-\infty$ to $+\infty$. Evaluating that integral gives, for any point in the midplane,

$$\mathbf{E}(r)=\frac{\lambda\hat{\mathbf{r}}}{2\pi\epsilon_0 r} \tag{2.15}$$

This is the same result that we found for a finite line charge for points near the line. From close enough the finite line looks, to a first approximation, infinitely long.

Example: What is the electric field on the axis of a circular loop of uniformly charged thin wire with total charge q ? Let a be the radius of the wire circle.

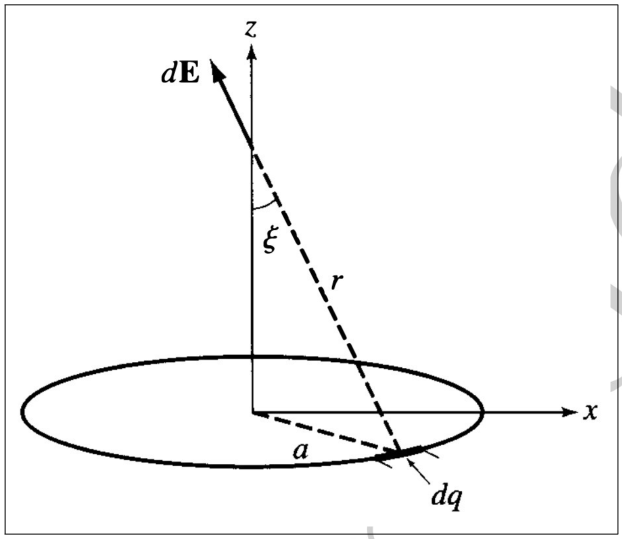


Figure 2.4: A charged wire. $(0,0,z')$ is a source point and $(x,0,0)$ is a field point.

In Figure 2.4 the wire is in the xy plane centered at O . The charge element dq produces field $d\mathbf{E}$ at the point $(0,0,z)$, as shown. Because the charge distribution is axially symmetric, the resultant field of the entire wire is in the z direction, and $dE_z = dE\cos\xi = dE\left(z/\sqrt{a^2+z^2}\right)$. Each charge element makes the same contribution to dE_z because they are all at the same distance $\sqrt{a^2+z^2}$ from the field point. Therefore the Electric field is

$$E_z(0,0,z) = \frac{qz}{4\pi\epsilon_0\left(a^2+z^2\right)^{3/2}} \tag{2.16}$$

A charged disk, or a charged plane, can be built up from elemental annuli, so (2.16) can be integrated over the loop radius to find the field of a disk or plane.

2.3 Gauss Law and applications

2.3.1 Flux

The notion of flux is an important one in physics. It refers to the flow of some vectorial quantity through an area. The simplest example to picture is the flow of fluid. Imagine that flowing with velocity $\vec{v} = v\hat{x}$. Now imagine that we dip a square wire loop of cross section area A into this fluid. We describe this loop with a vector $\vec{A} = A\hat{n}$, where $\hat{n}$ is the normal to the plane of the loop.

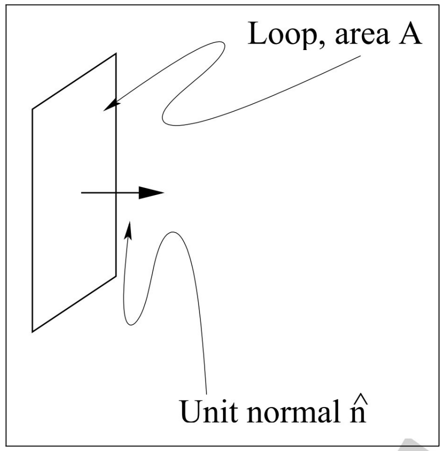


Figure 2.5: Unit normal of a loop of area A

The velocity flux Φ_v is defined as the overlap between the velocity vector $\vec{v}$ and the loop area $\vec{A}$:

$$\Phi_v = \vec{v} \cdot \vec{A} \tag{2.17}$$

For a uniform, plane loop, and constant, uniform velocity $\vec{v}$, we have $\Phi_v = vA\cos\theta$, where θ is the angle between $\hat{n}$ and the velocity vector:

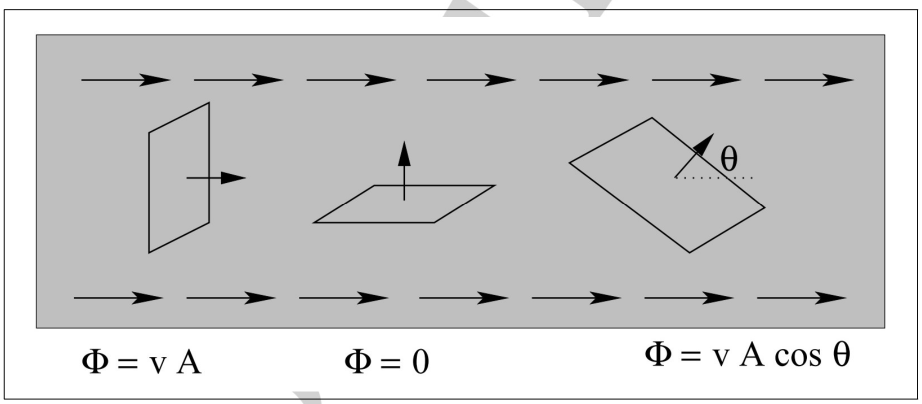


Figure 2.6: Flux

For this example, the rate of fluid flow through this loop is proportional to this flux, Φ_v .

What if things aren't nice and uniform like this? Then, we break the cross sectional area of the loop up into little squares $d\vec{A}$; we assign each square its own normal $\hat{n}$. The flux then becomes an integral:

$$\Phi_v = \int \vec{v} \cdot d\vec{A} \tag{2.18}$$

The integral is taken over the entire surface through which we wish to compute the flux.

2.3.2 Electric flux and Gauss Law

The electric flux Φ_E (which I'll usually just write Φ) is just like the velocity flux discussed above, using the electric field $\vec{E}$ rather than the fluid velocity $\vec{v}$:

$$\Phi = \int_{\text{Surface}} \vec{E} \cdot d\vec{A} \tag{2.19}$$

Electric flux has a very nice interpretation in terms of field lines. To motivate this interpretation, imagine that we have a uniform electric field. Let's calculate the flux through a square with cross sectional area A :

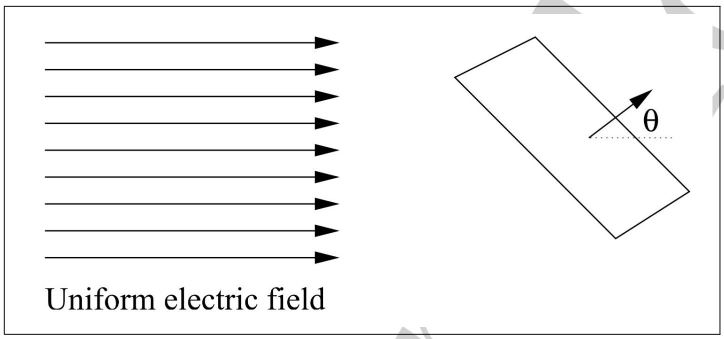


Figure 2.7: Flux of Electric Field

$$\Phi = (\vec{E}) \cdot (A\hat{n}) = EA\cos\theta \tag{2.20}$$

The Gauss which is just as equivalent as Coulomb's law is expressed as **the total electric flux through a closed surface is equals to the charge enclosed by the surface divided by the constant** ϵ_0

$$\oint_S \mathbf{E} \cdot d\mathbf{A} = \frac{1}{\epsilon_0} \int_V \rho(\mathbf{x}) d^3x = \frac{Q_{\text{ enclosed}}}{\epsilon_0} \tag{2.21}$$

Notice that it doesn't matter what shape the surface S takes. As long as it surrounds a total charge Q, the flux through the surface will always be Q/ϵ_0. This is shown, for example, in the left-hand figure below. The choice of S is called the Gaussian surface; often there's a smart choice that makes a particular problem simple.

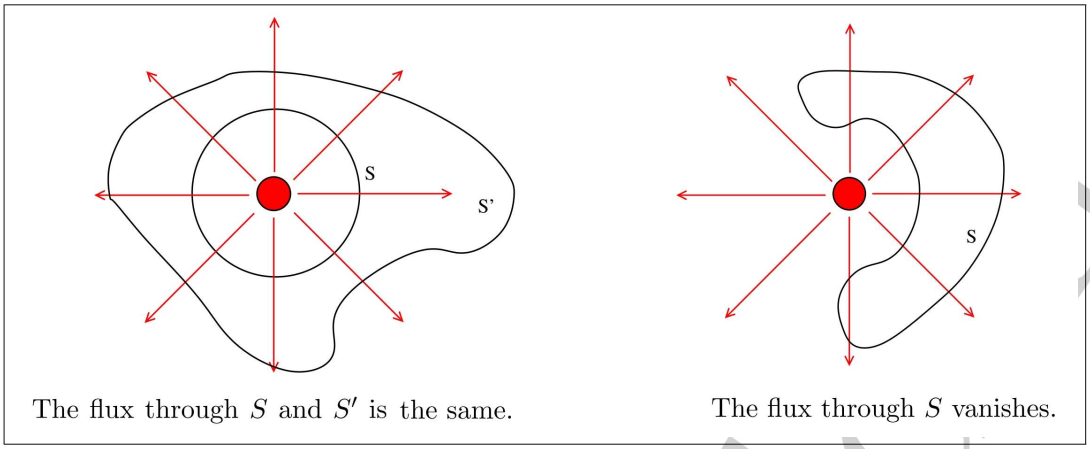


Figure 2.8: Flux of Electric Field

Only charges that lie inside V contribute to the flux. Any charges that lie outside will produce an electric field that penetrates through S at some point, giving negative flux, but leaves through the other side of S, depositing positive flux. The total contribution from these charges that lie outside of V is zero, as illustrated in the right-hand figure above.

2.3.3 Applications of Gauss Law

Solid Sphere: What will be the electric field inside and outside of a uniformly charged solid sphere of radius R and containing total charge Q?

Solution: Outside Look at the left side of the figure below. We want to know the electric field at some radius $r > R$. We take our Gaussian surface S to be a sphere of radius r as shown in the figure. Gauss' law states

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Q}{\epsilon_0}$$

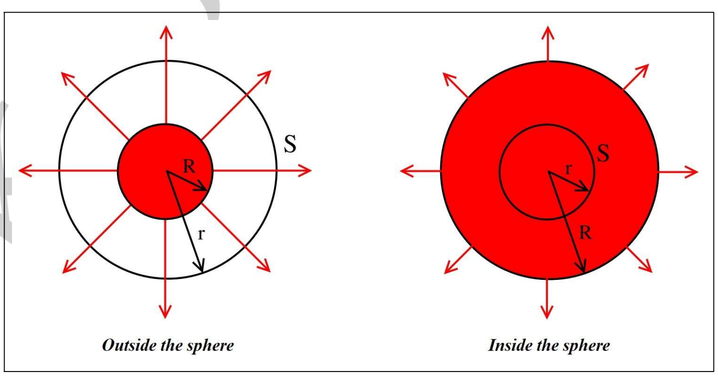


Figure 2.9: Electric Field inside and outside of a uniformly charged sphere

At this point we make use of the spherical symmetry of the problem. This tells us that the electric field must point radially outwards: $\mathbf{E}(\mathbf{x}) = E(r)\hat{\mathbf{r}}$. And, since the integral is only over the angular coordinates of the sphere, we can pull the function $E(r)$ outside. We have

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r) 4\pi r^2 = \frac{Q}{\epsilon_0}$$

where the factor of $4\pi r^2$ has arisen simply because it's the area of the Gaussian sphere. We learn that the electric field outside a spherically symmetric distribution of charge Q is

$$\mathbf{E}(\mathbf{x}) = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}} \tag{2.22}$$

Finally, note that the assumption of symmetry was crucial in our above analysis. Without it, the electric field $\mathbf{E}(\mathbf{x})$ would have depended on the angular coordinates of the sphere S and so been stuck inside the integral. In situations without symmetry, Gauss' law alone is not enough to determine the electric field and we need to also use $\nabla \times \mathbf{E} = 0$. If you're worried, however, it's simple to check that our final expression for the electric field in (2.22) does indeed solve $\nabla \times \mathbf{E} = 0$

Solution: Inside The electric field outside a spherically symmetric charge distribution is always given by (2.22) What about inside? Look at the right side of figure 2.9 This depends on the distribution in question. The simplest is a sphere of radius R with uniform charge distribution ρ. The total charge is

$$Q = \frac{4\pi}{3}R^3\rho$$

Let's pick our Gaussian surface to be a sphere, centered at the origin, of radius $r < R$. The charge contained within this sphere is $4\pi\rho r^3/3 = Qr^3/R^3$, so Gauss' law gives

$$\int_S \mathbf{E} \cdot d\mathbf{S} = \frac{Qr^3}{\epsilon_0 R^3}$$

Again, using the symmetry argument we can write $\mathbf{E}(\mathbf{r}) = E(r)\hat{\mathbf{r}}$ and compute

$$\int_S \mathbf{E} \cdot d\mathbf{S} = E(r) \int_S \hat{\mathbf{r}} \cdot d\mathbf{S} = E(r) 4\pi r^2 = \frac{Qr^3}{\epsilon_0 R^3}$$

This tells us that the electric field grows linearly inside the sphere

$$\mathbf{E}(\mathbf{x}) = \frac{Qr}{4\pi\epsilon_0 R^3}\hat{\mathbf{r}} \quad r < R \tag{2.23}$$

Outside the sphere we revert to the inverse-square form (2.22). At the surface of the sphere, $r = R$, the electric field is continuous but the derivative, dE/dr, is not. This is shown in the graph.

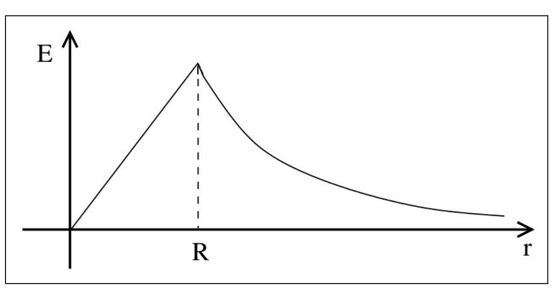
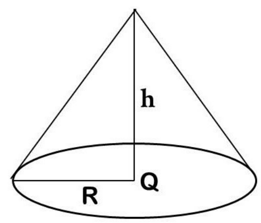


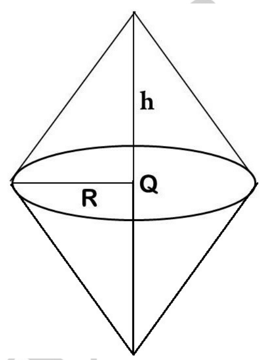
Figure 2.10: Graph of Electric Field inside and outside of a uniformly charged sphere

Example: Consider a charge Q at the origin of 3-dimensional coordinate system. The flux of the electric field through the curved surface of a cone that has a height h and a circular base of radius R (as shown in figure) is **[NET Dec 2015]**



(a) $\frac{Q}{\varepsilon_0}$ (b) $\frac{Q}{2\varepsilon_0}$ (c) $\frac{hQ}{R\varepsilon_0}$ (d) $\frac{QR}{2h\varepsilon_0}$

Solution: The Gauss law will be implemented in a clever manner. Imagine another cone base of which is placed in the base of the original cone and thus make the system closed. The system is now look like this



Now the total charge enclosed by the closed cone is Q. Hence the total flux is $\Phi = Q/\epsilon_0$. The flux passes through the whole surface. The syatem is symmetric as is composed of exactly two cone. The flux passes through the upper cone and through the lower cone is same. Hence the flus passes through the curved surface of the upper cone is $\Phi/2 = \frac{Q}{2\varepsilon_0}$

Problem: A sphere of radius R_1 has charge density ρ uniform within its volume, except for a small spherical hollow region of radius R_2 located a distance a from the center.

Find the field **E** at the center of the hollow sphere.

Solution: Consider an arbitrary point P of the hollow region as shown in figure 2.11 and let

$$\mathbf{OP} = \mathbf{r}, \quad \mathbf{Q'P} = \mathbf{r}', \quad \mathbf{OO'} = \mathbf{a}, \quad \mathbf{r}' = \mathbf{r} - \mathbf{a}$$

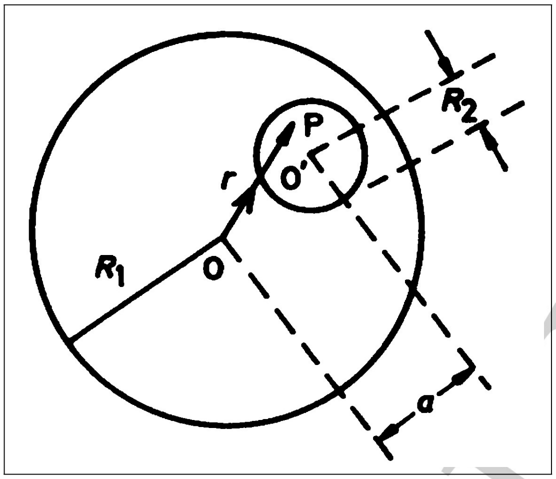


Figure 2.11: Electric field inside hole

If there were no hollow region inside the sphere, the electric field at the point P would be

$$\mathbf{E}_1 = \frac{\rho}{3\varepsilon_0}\mathbf{r}$$

If only the spherical hollow region has charge density ρ the electric field at P would be

$$\mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{r}'$$

Hence the superposition theorem gives the electric field at P as

$$\mathbf{E} = \mathbf{E}_1 - \mathbf{E}_2 = \frac{\rho}{3\varepsilon_0}\mathbf{a}$$

Thus the field inside the hollow region is uniform. This of course includes the center of the hollow.

There are many examples in the standard books like Griffiths. Please do some exercises from Griffiths also.

2.4 Divergence and Curl of Electric Field

Divergence and curl of Electric field is given by (Remember, here we are dealing with the electrostatic case, when the charges of the system are not moving. In the dynamic case the curl will be different)

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} \tag{2.24}$$

$$\nabla \times \mathbf{E} = 0 \tag{2.25}$$

We write $\mathbf{E}(\mathbf{x})$ in terms of a scalar function $V(\mathbf{x})$ as

$$\mathbf{E}(\mathbf{x}) = -\nabla V(\mathbf{x}) \tag{2.26}$$

The very important function $V(\mathbf{x})$ is called the electric potential.

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho\left(\mathbf{x}'\right) d^3x'}{\left|\mathbf{x}-\mathbf{x}'\right|} \tag{2.27}$$

Applying it with the divergence of electric field we get

$$-\nabla^2 V = \rho/\epsilon_0 \tag{2.28}$$

This is called Poisson's equation. In regions of space where the charge density vanishes, we're left solving the Laplace equation

$$\nabla^2 V = 0 \tag{2.29}$$

2.5 Electric Potential

It is instructive to construct $V(\mathbf{x})$ in another way. Let $-V(\mathbf{x})$ be the line integral of $\mathbf{E}$ along a curve Γ from a reference point $\mathbf{x}_0$ to $\mathbf{x}$

$$V(\mathbf{x}) = -\int_\Gamma \mathbf{E}\cdot d\ell \tag{2.30}$$

This equation says that $V(\mathbf{x})$ is equal to the work that must be done by an external force, acting against the electric force, to move unit charge from $\mathbf{x}_0$ to $\mathbf{x}$ along the path Γ, because $-\mathbf{E}$ is the force per unit charge that the external agent must exert.

Example: The electric potential of some configuration is given by the expression

$$V(\mathbf{r}) = A\frac{e^{-\lambda r}}{r}$$

where A and λ are constants. Find the electric field $\mathbf{E}(\mathbf{r})$, the charge density $\rho(r)$, and the total charge Q . **Solution:**

$$\mathbf{E} = -\nabla V = -A\frac{\partial}{\partial r}\left(\frac{e^{-\lambda r}}{r}\right)\hat{\mathbf{r}} = -A\left\{\frac{r(-\lambda)e^{-\lambda r} - e^{-\lambda r}}{r^2}\right\} = Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

So the potential

$$\rho = \epsilon_0 \boldsymbol{\nabla}\cdot\mathbf{E} = \epsilon_0 A\left\{e^{-\lambda r}(1+\lambda r)\boldsymbol{\nabla}\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) + \frac{\hat{\mathbf{r}}}{r^2}\cdot\boldsymbol{\nabla}\left(e^{-\lambda r}(1+\lambda r)\right)\right\}$$

But as you know

$$\nabla\cdot\left(\frac{\hat{\mathbf{r}}}{r^2}\right) = 4\pi\delta^3(\mathbf{r})$$

And from the properties of the delta function

$$e^{-\lambda r}(1+\lambda r)\delta^3(\mathbf{r}) = \delta^3(\mathbf{r})$$

The gradient

$$\nabla\left(e^{-\lambda r}(1+\lambda r)\right) = \hat{\mathbf{r}}\frac{\partial}{\partial r}\left(e^{-\lambda r}(1+\lambda r)\right)$$

$$= \hat{\mathbf{r}} \left\{ -\lambda e^{-\lambda r}(1+\lambda r) + e^{-\lambda r}\lambda \right\} = \hat{\mathbf{r}} \left(-\lambda^2 r e^{-\lambda r} \right)$$

So we get

$$\frac{\hat{\mathbf{r}}}{r^2} \cdot \nabla \left(e^{-\lambda r}(1+\lambda r) \right) = -\frac{\lambda^2}{r} e^{-\lambda r}$$

And hence the charge density

$$\rho = \epsilon_0 A \left[4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r} e^{-\lambda r} \right]$$

Hence the total charge

$$\begin{aligned} Q = \int \rho d\tau &= \epsilon_0 A \left\{ 4\pi \int \delta^3(\mathbf{r}) d\tau - \lambda^2 \int \frac{e^{-\lambda r}}{r} 4\pi r^2 dr \right\} \\ &= \epsilon_0 A \left(4\pi - \lambda^2 4\pi \int_0^\infty r e^{-\lambda r} dr \right) \\ &= 4\pi\epsilon_0 A \left(1 - \frac{\lambda^2}{\lambda^2} \right) = 0 \end{aligned}$$

2.6 Conductors

Let's now throw something new into the mix. A conductor is a region of space which contains charges that are free to move. Physically, think "metal". We want to ask what happens to the story of electrostatics in the presence of a conductor. There are a number of things that we can say straight away:

- Inside a conductor we must have $\mathbf{E} = 0$. If this isn't the case, the charges would move. But we're interested in electrostatic situations where nothing moves.
- since $\mathbf{E} = 0$ inside a conductor, the electrostatic potential ϕ must be constant throughout the conductor.
- since $\mathbf{E} = 0$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$, we must also have $\rho = 0$. This means that the interior of the conductor can't carry any charge.
- Conductors can be neutral, carrying both positive and negative charges which balance out. Alternatively, conductors can have net charge. In this case, any net charge must reside at the surface of the conductor.
- since ϕ is constant, the surface of the conductor must be an equipotential. This means that any $\mathbf{E} = -\nabla\phi$ is perpendicular to the surface. This also fits nicely with the discussion above since any component of the electric field that lies tangential to the surface would make the surface charges move.
- If there is surface charge σ anywhere in the conductor then, from the boundary condition of the electric field, together with the fact that $\mathbf{E} = 0$ inside, the electric field just outside the conductor must be

$$\mathbf{E} = \frac{\sigma}{\epsilon_0} \hat{\mathbf{n}}$$

Problems involving conductors are of a slightly different nature than those we've discussed up to now. The reason is that we don't know from the start where the charges are, so we don't know what charge distribution ρ that we should be solving for. Instead, the electric fields from other sources will cause the charges inside the conductor to shift around until they reach equilibrium in such a way that $E = 0$ inside the conductor. In general, this will mean that even neutral conductors end up with some surface charge, negative in some areas, positive in others, just enough to generate an electric field inside the conductor that precisely cancels that due to external sources.

Example: A Conducting Sphere

To illustrate the kind of problem that we have to deal with, it's probably best just to give an example. Consider a constant background electric field. Now place a neutral, spherical conductor inside this field. What happens?

We know that the conductor can't suffer an electric field inside it. Instead, the mobile charges in the conductor will move: the negative ones to one side; the positive ones to the other. The sphere now becomes polarised. These charges counteract the background electric field such that $\mathbf{E} = 0$ inside the conductor, while the electric field outside impinges on the sphere at right-angles. The end result must look qualitatively like this:

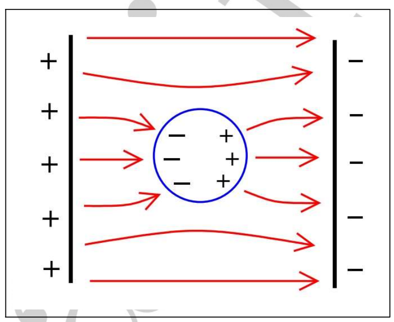


Figure 2.12: Conducting sphere in an Electric field

We'll learn how to compute the electric field in this, and related, situations later.

2.7 Image Problem

For particularly simple situations, there is a rather cute method that we can use to solve problems involving conductors. Although this technique is somewhat limited, it does give us some good intuition for what's going on. It's called the method of images.

Example: A charged particle near a conducting plane: Suppose a point charge q is held a distance d above an infinite grounded conducting plane. What is the potential in the region above the plane? It's not just $(1/4\pi\epsilon_0)\, q/n$, for q will induce a certain amount of negative charge on the nearby surface of the conductor; the total potential is due in part to q directly, and in part to this induced charge. But how can we possibly calculate the potential, when we don't know how much charge is induced or how it is distributed?

We're looking for a solution to the Poisson equation with a delta-function source at z = d = $(0, 0, d)$, together with the requirement that $\phi = 0$ on the plane $z = 0$. From the uniqueness theorem, there's a unique solution to this kind of problem. We just have to find it.

Here's the clever trick. Forget that there's a conductor at $z < 0$. Instead, suppose that there's a charge $-q$ placed opposite the real charge at $z = -d$. This is called the image charge. The potential for this pair of charges is just the potential

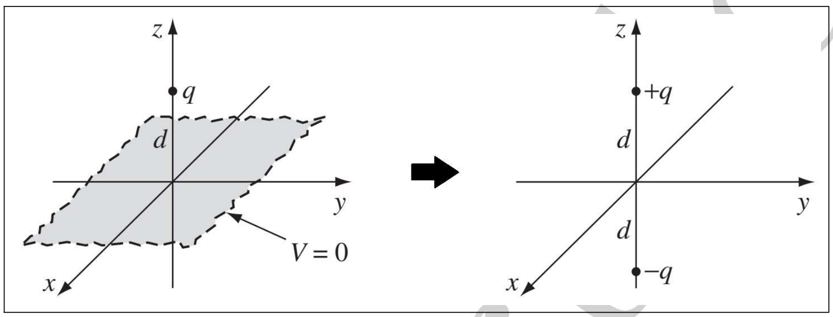


Figure 2.13: Image problem: Charge in front of grounded plane conductor

$$V(x, y, z) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{x^2 + y^2 + (z - d)^2}} - \frac{q}{\sqrt{x^2 + y^2 + (z + d)^2}}\right] \quad (2.31)$$

(The denominators represent the distances from (x, y, z) to the charges $+q$ and $-q$, respectively.) It follows that

(1) $V = 0$ when $z = 0$ and (2) $V \to 0$ for $x^2 + y^2 + z^2 \gg d^2$ (3) The only charge in the region $z > 0$ is the point charge $+q$ at $(0, 0, d)$.

But these are precisely the conditions of the original problem! Evidently the second configuration happens to produce exactly the same potential as the first configuration, in the "upper" region $z \geq 0$. Conclusion: The potential of a point charge above an infinite grounded conductor is given by (2.31) for $z \geq 0$.

What happen in The "lower" region, $z < 0$, ? As there are a conductor in $z = 0$, which restricts any influence of the charge at $z > 0$ to penetrate into $z < 0$ region, the field in $z < 0$ remains ZERO.

The electric field in the z direction

$$E_z = -\frac{\partial V}{\partial x} = \frac{q}{4\pi\epsilon_0}\left(\frac{z - d}{|\mathbf{r} - \mathbf{d}|^{3/2}} - \frac{z + d}{|\mathbf{r} + \mathbf{d}|^{3/2}}\right) \quad x \geq 0 \quad (2.32)$$

Meanwhile, $E_z = 0$ for $z < 0$. The discontinuity of E_z at the surface of the conductor determines the induced surface charge . The value is

$$\sigma == -\epsilon_0 \frac{\partial V}{\partial n} = -\epsilon_0 \left. \frac{\partial V}{\partial z} \right|_{z=0} = E_z \left. \epsilon_0 \right|_{z=0}$$

As

$$\frac{\partial V}{\partial z} = \frac{1}{4\pi\epsilon_0} \left\{ \frac{-q(z-d)}{\left[x^2+y^2+(z-d)^2\right]^{3/2}} + \frac{q(z+d)}{\left[x^2+y^2+(z+d)^2\right]^{3/2}} \right\}$$

So the charge density

$$\sigma(x,y) = \frac{-qd}{2\pi \left(x^2+y^2+d^2\right)^{3/2}}$$

The total induced charge

$$Q = \int \sigma da$$

$$Q = \int_0^{2\pi} \int_0^{\infty} \frac{-qd}{2\pi \left(r^2+d^2\right)^{3/2}} r dr d\phi = \left. \frac{qd}{\sqrt{r^2+d^2}} \right|_0^{\infty} = -q$$

The total charge induced on the plane is $-q$, as expected.

The force on the charge is

$$\mathbf{F} = -\frac{1}{4\pi\epsilon_0} \frac{q^2}{(2d)^2} \hat{\mathbf{z}}$$

This force is attractive, pulling the charge towards the conductor.

Example: A charged particle near a conducting sphere: A point charge q is situated a distance a from the center of a grounded conducting sphere of radius R . Find the potential outside the sphere.

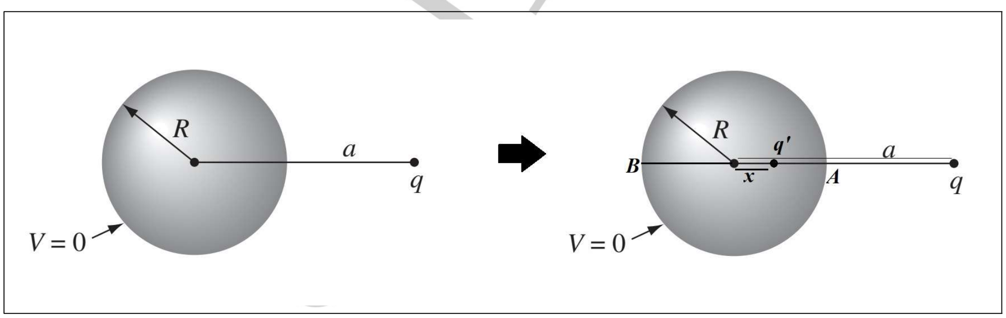


Figure 2.14: Image problem: Charge in front of grounded Spherical conductor

You can assume a image charge q' at x distance from the center along the axis in which the q charge has been placed as shown in the right side of the figure. Now to potential has to be zero everywhere at the surface of the sphere. Evaluate the potential at point A and B as shown in the figure.

$$V_A = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{a-R} + \frac{q'}{R-x} \right) = 0$$

and

$$V_B = \frac{1}{4\pi\epsilon_0} \left(\frac{q}{a+R} + \frac{q'}{R+x} \right) = 0$$

From these two equations you get

$$q' = -\frac{R}{a}q; \qquad x = \frac{R^2}{a}$$

If you take the line on which the charge q is placed to be your z axis and the position of q is, then $(0,0,d)$ the potential at some point (x,y,z) is now

$$V(X,Y,Z) = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{\sqrt{(x^2+y^2+(z-d)^2}} - \frac{R}{d}\frac{1}{\sqrt{x^2+y^2+(z-R^2/d)^2}}\right) \tag{2.33}$$

Using the laws of cosines

$$\mathscr{r}^{\,2} = r^2 + a^2 + 2ra\cos\theta$$

and

$$\mathscr{r}^{\,\prime 2} = r^2 + \left(\frac{R^2}{a}\right)^2 + 2r\frac{R^2}{a}\cos\theta$$

the potential can be written by as shown in the figure

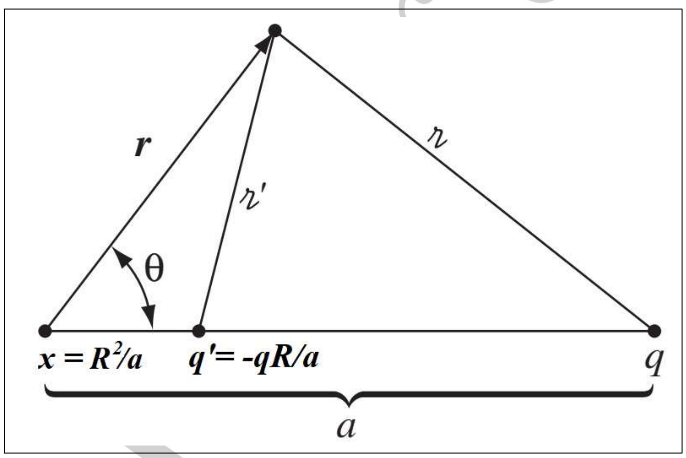


Figure 2.15: Charge in front of grounded Spherical conductor: Calculation of potential

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+a^2-2ra\cos\theta}} - \frac{q}{\sqrt{R^2+(ra/R)^2-2ra\cos\theta}}\right] \tag{2.34}$$

Example: A hollow metallic sphere of radius a, which is kept at a potential V_0 has a charge Q at its centre. The potential at a point outside the sphere, at a distance r from the centre, is **[NET Dec 2015]**

(a) V_0 (b) $\frac{Q}{4\pi\epsilon_0 r} + \frac{V_0 a}{r}$ (c) $\frac{Q}{4\pi\epsilon_0 r} + \frac{V_0 a^2}{r^2}$ (d) $\frac{V_0 a}{r}$

Solution: The most important thing is to consider the boundary condition. The potential is fixed at V_0 at radius a, so whatever be the charges present inside or outside the sphere, the potential remains constant.

How much charge we need to keep to make the potential V_0 at $r = a$? Obviously if we keep charge Q_0 at the center, the potential is is $\frac{Q_0}{4\pi\epsilon_0 a}$ at the radius. So

$$V_0 = \frac{Q_0}{4\pi\epsilon_0 a}; \quad \Longrightarrow \quad Q_0 = 4\pi\epsilon_0 a V_0$$

That is the image charge. it will produce the potential

$$V(r) = \frac{Q_0}{4\pi\epsilon_0 r} = \frac{V_0 a}{r}$$

outside the sphere.

Now what is the effect the charge Q kept at the center? Well, it won't change anything as long as the boundary condition remains same. that is the essence of uniqueness theorem, right? If you change the charge distribution but remains the boundary conditions same, then the potential won't change. Only the image charges redistribute themselves. So the charge Q don't effect the potential outside.

If you wonder what is the image charge now? Well the total charge at the center remains to be $4\pi\epsilon_0 a V_0$. So the induced charge is $Q' = 4\pi\epsilon_0 a V_0 - Q$.

2.8 Multopole expansion of electric field

The potential and field due to a point charge at the origin are

$$V(\mathbf{x}) = \frac{q}{4\pi\epsilon_0 r} \quad \text{and} \quad \mathbf{E}(\mathbf{x}) = \frac{q\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2}$$

Here $r = |\mathbf{x}|$ and $\hat{\mathbf{r}} = \mathbf{x}/r$. Now consider a collection of charges in a region near the origin. The problem we shall solve in this section is to determine the aymptotic form of $V(\mathbf{x})$, i.e., far from the charge distribution. The complete treatment of this problem is to express $V(\mathbf{x})$ as an expansion in powers of $1/r$, called the multipole expansion. We won't derive the full multipole expansion, but rather consider the first few terms, which are the most important.

As a first step, consider just two charges, q_1 at $\mathbf{x}_1$ and q_2 at $\mathbf{x}_2$. The exact potential function is the superposition

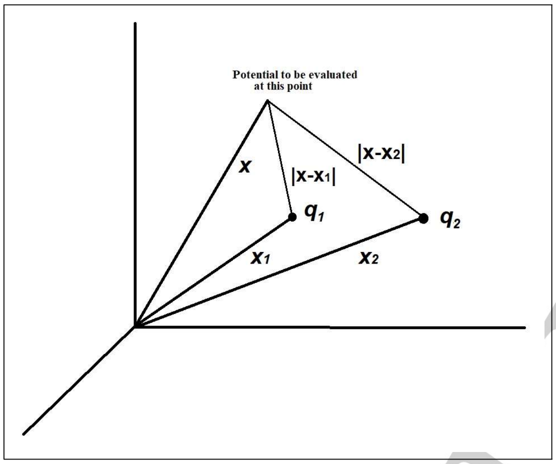


Figure 2.16: Vector representation of potential of two charge

$$V(\mathbf{x}) = \frac{q_1}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}_1|} + \frac{q_2}{4\pi\epsilon_0 |\mathbf{x} - \mathbf{x}_2|} \tag{2.35}$$

Asymptotically, i.e., for $r \gg r_1$ and $r \gg r_2$, we may make the expansion (for $k = 1$ or 2)

$$\frac{1}{|\mathbf{x} - \mathbf{x}_k|} = \frac{1}{\sqrt{r^2 - 2rr_k\cos\theta + r_k^2}} = \frac{1}{r\sqrt{1+\epsilon}} \tag{2.36}$$

where $\epsilon = \left(-2rr_k\cos\theta + r_k^2\right)/r^2$ and θ is the angle between $\mathbf{x}$ and $\mathbf{x}_k$. Expand the result in ϵ using the Taylor series

$$\frac{1}{\sqrt{1+\epsilon}} = 1 - \frac{1}{2}\epsilon + \frac{3}{8}\epsilon^2 + \mathcal{O}\left(\epsilon^3\right) \tag{2.37}$$

and reexpress the result in powers of $1/r$, dropping terms of order r_k^3/r^4. We obtain the result

$$\frac{1}{|\mathbf{x} - \mathbf{x}_k|} = \frac{1}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{x}_k}{r^2} + \frac{3\left(\hat{\mathbf{r}}\cdot\mathbf{x}_k\right)^2 - r_k^2}{2r^3} + \mathcal{O}\left(\frac{1}{r^4}\right) \tag{2.38}$$

multipole expansion for $V(\mathbf{x})$ by substituting these results and neglecting terms higher than quadrapoles we get (try to do by yourself, if you get frustrated, leave it)

$$V(\mathbf{x}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{Q}{r} + \frac{\hat{\mathbf{r}}\cdot\mathbf{p}}{r^2} + \frac{\hat{\mathbf{r}}\cdot\mathcal{Q}_2\cdot\hat{\mathbf{r}}}{r^3}\right\} \tag{2.39}$$

The three terms are called the monopole (Q), dipole (p), and quadupole ($\mathcal{Q}_2$) terms. Q is a scalar, equal to $q_1 + q_2$, the total charge. The monopole term is dominant as $r \to \infty$ unless $Q = 0$. The parameter $\mathbf{p}$ is a vector, given by

$$\mathbf{p} = q_1\mathbf{x}_1 + q_2\mathbf{x}_2 \tag{2.40}$$

called the electric dipole moment of the system. The dipole term is dominant at large r if $Q = 0$. The parameter $\mathcal{Q}_2$ is a tensor , given by

$$\mathcal{Q}_2 = \frac{q_1}{2}\left[3\mathbf{x}_1\mathbf{x}_1 - r_1^2\mathbf{I}\right] + \frac{q_2}{2}\left[3\mathbf{x}_2\mathbf{x}_2 - r_2^2\mathbf{I}\right] \tag{2.41}$$

called the quadrupole moment. Here I denotes the unit tensor. As a vector has one index, a tensor has two indices; for example, $(\mathbf{x}_1\mathbf{x}_1)_{ij} = x_{1i}x_1j$ and $(\mathbf{I})_{ij} = \delta_{ij}$

Monopole, dipole and quadrapole moment for discrete (individual point charges) charge distribution is

$$\begin{aligned} Q &= \sum_{k=1}^{N} q_k \\ \mathbf{p} &= \sum_{k=1}^{N} q_k \mathbf{x}_k \\ \mathcal{Q}_2 &= \sum_{k=1}^{N} \frac{q_k}{2}\left(3\mathbf{x}_k\mathbf{x}_k - r_k^2\mathbf{I}\right) \end{aligned} \tag{2.42}$$

Monopole, dipole and quadrapole moment for continuous charge distribution is

$$\begin{aligned} Q &= \int \rho\left(\mathbf{x}'\right) d^3x' \\ \mathbf{p} &= \int \mathbf{x}'\rho\left(\mathbf{x}'\right) d^3x' \\ \mathcal{Q}_2 &= \int \frac{1}{2}\left(3\mathbf{x}'\mathbf{x}' - r'^2\mathbf{I}\right)\rho\left(\mathbf{x}'\right) d^3x' \end{aligned} \tag{2.43}$$

Notice the dependence on r for the various multipoles: For a charge (monopole) the potential decreases as $1/r$ and the field as $1/r^2$; for a dipole the potential de- creases as $1/r^2$ and the field as $1/r^3$; for a quadrupole the potential decreases as as $1/r^3$ and the field as $1/r^4$. This pattern generalizes to higher multipoles.

Example: To understand the notations we see the example. Charges q_1 and q_2 on the z axis, at $z = d_1$ and $z = d_2$ respectively, as shown in figure.

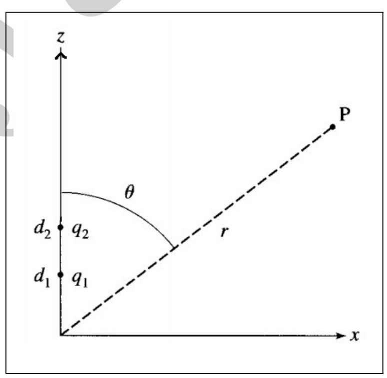


Figure 2.17: Two charges q_1 and q_2: Calculation of potential

By symmetry, $V(\mathbf{x})$ is $V(r,\theta)$ (independent of ϕ). The exact potential is

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_1}{\sqrt{r^2 - 2rd_1\cos\theta + d_1^2}} + \frac{q_2}{\sqrt{r^2 - 2rd_2\cos\theta + d_2^2}}\right\} \tag{2.44}$$

The multipole expansion through order r^{-3}, valid for $r \gg d_1, d_2$, is

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left\{\frac{q_1+q_2}{r} + \frac{(q_1d_1+q_2d_2)\cos\theta}{r^2} + \frac{(q_1d_1^2+q_2d_2^2)}{2r^3}\left(3\cos^2\theta - 1\right)\right\} \tag{2.45}$$

This result agrees with the general equation (2.39). The total charge is $Q = q_1 + q_2$, and the dipole moment is $\mathbf{p} = (q_1d_1 + q_2d_2)\,\hat{k}$. The quadrupole moment is

$$\mathcal{Q}_2 = \frac{1}{2}\left(q_1d_1^2 + q_2d_2^2\right)(3\hat{k}\hat{k} - \mathbf{I}) \tag{2.46}$$

In general the dipole and quadrupole moments depend on the choice of origin. For example, (2.45) shows that the multipole expansion for a single charge displaced from the origin, e.g., q_1 at $(0,0,d_1)$ with $q_2 = 0$, has nonzero dipole and quadrupole moments, $\mathbf{p} = q_1d_1\hat{k}$ and $\mathcal{Q}_2 = q_1d_1^2(3\hat{k}\hat{k} - \mathbf{I})/2$; whereas if the origin were chosen to be the position of q_1 there would be only a monopole term. An interesting special case occurs if the total charge in the distribution is 0 ; then. p does not depend on the choice of origin.

2.8.1 Electric Dipole

For two equal but opposite charges, i.e., $q_1 = q$ and $q_2 = -q$, the total charge is $Q = 0$, and the dipole moment is

$$\mathbf{p} = q\mathbf{d} \tag{2.47}$$

where $\mathbf{d} = \mathbf{x}_1 - \mathbf{x}_2$ is the vector from the negative charge $(q_2 = -q)$ to the positive charge $(q_1 = q)$. This system is called an electric dipole. An important limiting case is the limit $d \to 0$ with $\mathbf{p}$ fixed, called a point-like electric dipole. In this limit, and taking the origin to be the position of the dipole, all other multipole moments (quadrupole and higher) are 0. The potential of a point-like dipole is

$$V(\mathbf{x}) = \frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{4\pi\epsilon_0 r^2} \tag{2.48}$$

for all $\mathbf{x}$. If the dipole points in the z direction, i.e., $\mathbf{p} = p_0\hat{\mathbf{k}}$, then in spherical coordinates the dipole potential is

$$V(r,\theta) = \frac{p_0\cos\theta}{4\pi\epsilon_0 r^2} \tag{2.49}$$

The point-dipole potential is a good approximation for a neutral charge distribution that is small compared to all length scales in the problem. For example, molecule, having a size of order 10^{-10}m, acts as a point-like dipole when placed in a laboratory-scale field. The permanent dipole moments of small molecules such as H_2O, NH_3, HCl , or CO , are of the order of 1 debye (D), where $1\mathrm{D} = 3.33 \times 10^{-30}\mathrm{Cm}$. One debye is approximately the dipole moment of two charges $\pm 0.2e$ separated by 1 angstrom.

The electric field produced by a point-like electric dipole located at the origin is

$$\mathbf{E}(\mathbf{x}) = -\nabla V = \frac{3\hat{\mathbf{r}}(\mathbf{p}\cdot\hat{\mathbf{r}}) - \mathbf{p}}{4\pi\epsilon_0 r^3} \tag{2.50}$$

For the special case of a dipole $\mathbf{p} = p_0\hat{\mathbf{k}}$ located at the origin and pointing in the z direction, the field in spherical polar coordinates is

$$\mathbf{E}(\mathbf{x}) = -\nabla V = \frac{p_0}{4\pi\epsilon_0 r^3}(2\cos\theta\hat{\mathbf{r}} + \sin\theta\hat{\boldsymbol{\theta}}) \tag{2.51}$$

Example: A sphere of radius R centered at the origin, has charge density

$$\rho(r,\theta) = k\frac{R}{r^2}(R-2r)\sin\theta$$

where k is constant. Find the approximate potential outside the sphere along z-axis, far from the sphere.

Solution: We will expand the potential in multipoles. The monopole terms is zero, as can be seen by calculating the total charge,

$$\begin{aligned} Q =& kR\int_0^R dr\int_0^\pi \frac{R-2r}{r^2}\sin\theta 2\pi r^2\sin\theta d\theta \\ &= 2\pi kR\int_0^R (R-2r)dr\int_0^\pi \sin^2\theta d\theta = 2\pi kR\left(Rr - r^2\right)\Big|_0^R \times \frac{\pi}{2} = 0 \end{aligned}$$

We next calculate the dipole term, which is obtained by multiplying the integrand above by $r\cos\theta$. In this case, the angle integration is

$$\int_0^\pi \sin^2\theta\cos\theta d\theta = \left.\frac{\sin^3\theta}{3}\right|_0^\pi = 0$$

Thus the dipole term also does not contribute to the potential. We next calculate the quadrupole term, given by $Q = \int \rho(r)r^2\left(3\cos^2\theta - 1\right)d^3r$. This can be calculated as follows:

$$\begin{aligned} Q_{zz} &= 2\pi kR\int_0^R r^2(R-2r)dr\int_0^\pi \sin^2\theta\left(3\cos^2\theta - 1\right)d\theta \\ &= 2\pi kR\left(\frac{Rr^3}{3} - 2\frac{r^4}{4}\right)\Bigg|_0^R \times \int_0^\pi \left(3\cos^2\theta - 1\right)\sin^2\theta d\theta \\ &= 2\pi kR\frac{R^4}{6}\times\frac{\pi}{8} = \frac{k\pi^2R^5}{24} \end{aligned}$$

The potential due to the quadrupole is

$$V = \frac{1}{4\pi\epsilon_0}\frac{Q_{zz}}{2r^3} = \frac{k\pi R^5}{192\epsilon_0 r^3}$$

2.8.2 Torque and Force on a Dipole in an Electric and field

What happen when an electric dipole is placed in an electric field? It experiences some torque and force.

A dipole $\mathbf{p} = q\mathbf{d}$ in a uniform field $\mathbf{E}$ experiences a torque

$$\mathbf{N} = \mathbf{p} \times \mathbf{E} \tag{2.52}$$

Force on a dipole while placed in an non uniform (inhomogeneous) electric field

$$\mathbf{F} = q(\Delta\mathbf{E}) = (\mathbf{p} \cdot \nabla)\mathbf{E} \tag{2.53}$$

energy of an ideal dipole $\mathbf{p}$ in an electric field $\mathbf{E}$ is

$$U = -\mathbf{p} \cdot \mathbf{E} \tag{2.54}$$

This cn also be written as

$$U = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\mathbf{p}_1 \cdot \mathbf{p}_2 - 3\left(\mathbf{p}_1 \cdot \hat{\mathbf{r}}\right)\left(\mathbf{p}_2 \cdot \hat{\mathbf{r}}\right)\right] \tag{2.55}$$

3 Dielectrics

3.1 Polarization

3.1.1 Atomic polarizibility

What happens to a neutral atom when it is placed in an electric field $\mathbf{E}$? The atom got a tiny induced dipole moment $\mathbf{p}$, which points in the same direction as $\mathbf{E}$. Typically, this induced dipole moment is approximately proportional to the field (as long as the latter is not too strong):

$$\mathbf{p} = \alpha\mathbf{E} \tag{3.1}$$

The constant of proportionality α is called atomic polarizability. Its value depends on the detailed structure of the atom in question.

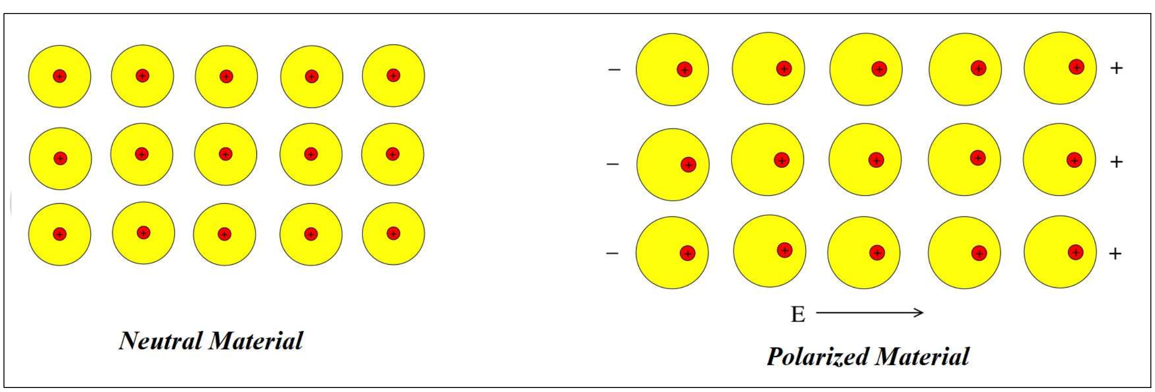


Figure 3.1: Neutral and polarized material

If the atom has radius a then the polarizibilty can be proved to be

$$\alpha = 4\pi\epsilon_0 a^3 = 3\epsilon_0 v \tag{3.2}$$

For non spherical atoms and molecules the polarizibility becomes a tensor. The relation now can be written as

$$\begin{aligned} p_x &= \alpha_{xx}E_x + \alpha_{xy}E_y + \alpha_{xz}E_z \\ p_y &= \alpha_{yx}E_x + \alpha_{yy}E_y + \alpha_{yz}E_z \\ p_z &= \alpha_{zx}E_x + \alpha_{zy}E_y + \alpha_{zz}E_z \end{aligned} \tag{3.3}$$

3.1.2 Polarization vector

Polarization vector is defined as

$$\mathbf{P} \equiv \text{ dipole moment per unit volume}$$

3.2 Bound charges

Whenever there is a polarization there will be some induced charges in the system. The Induced charges can be modelled as bound surface charges and bound volume charges

$$\rho_b = -\nabla.\mathbf{P} \tag{3.4}$$

$$\sigma_b = \mathbf{P}.\hat{n} \tag{3.5}$$

Example: A sphere of radius R carries a polarization

$$\mathbf{P}(\mathbf{r}) = k\mathbf{r}$$

where k is a constant and $\mathbf{r}$ is the vector from the center.

(a) Calculate the bound charges σ_b and ρ_b.
(b) Find the field inside and outside the sphere.

Solution: (a)

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = kR;$$

$$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 kr\right) = -\frac{1}{r^2}3kr^2 = -3k$$

(b) as there are no free charges the field will be from bound changes only. for $r < R$ the surface bound charge won't count as σ_b is at surface only.

$$r < R, \quad \mathbf{E} = \frac{1}{3\epsilon_0}\rho r\hat{\mathbf{r}} \quad \text{so} \quad \mathbf{E} = -\left(k/\epsilon_0\right)\mathbf{r}$$

For $r > R$, the contribution from both the surface and volume bound charges need to be calculated

$$Q_{\text{ tot }} = (kR)\left(4\pi R^2\right) + (-3k)\left(\frac{4}{3}\pi R^3\right) = 0,$$

so

$$\mathbf{E} = 0$$

One important result you need to remember; the electric field produced by a uniformly polarized sphere of radius R is

Inside:

$$\mathbf{E} = -\frac{1}{3\epsilon_0}\mathbf{P}, \quad \text{for } r < R$$

Outside: the potential and field is just like a perfect dipole at the origin

$$V = \frac{1}{4\pi\epsilon_0}\frac{\mathbf{p}\cdot\hat{\mathbf{r}}}{r^2}, \quad \text{for} \quad r \geq R$$

3.3 Electric Displacement Vector

We learned that the electric field obeys Gauss' law

$$\vec{\nabla}.\vec{E} = \rho/\epsilon_0$$

This is a fundamental law of Nature. It doesn't change just because we're inside a material. But, from our discussion above, we see that there's a natural way to separate the electric charge into two different types. There is the bound charge ρ_b that arises due to polarisation. And then there is anything else. This could be some electric impurities that are stuck in the dielectric, or it could be charge that is free to move because our insulator wasn't quite as good an insulator as we originally assumed. The only important thing is that this other charge does not arise due to polarisation. We call this extra charge free charge, ρ_f .

Within the dielectric, the total charge density is

$$\rho = \rho_b + \rho_f \tag{3.6}$$

and we know $\rho_b = -\vec{\nabla}.\vec{P}$, hence

$$\epsilon_0\nabla\cdot\mathbf{E} = \rho = \rho_b + \rho_f = -\nabla\cdot\mathbf{P} + \rho_f \tag{3.7}$$

we write the equation as

$$\nabla\cdot\left(\epsilon_0\mathbf{E} + \mathbf{P}\right) = \rho_f \tag{3.8}$$

Now define

$$\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P} \tag{3.9}$$

In integral form

$$\oint \mathbf{D} \cdot d\mathbf{a} = Q_{f_{\text{enc}}} \tag{3.10}$$

where $Q_{f_{\text{enc}}}$ denotes the total free charge enclosed in the volume.

That's quite nice. Gauss' law for the displacement involves only the free charge; any bound charge arising from polarisation has been absorbed into the definition of $\mathbf{D}$

3.3.1 Linear Dielectrics

In general, the polarisation $\mathbf{P}$ can be a complicated function of the electric field $\mathbf{E}$. However, most materials it turns out that $\mathbf{P}$ is proportional to $\mathbf{E}$. Such materials are called linear dielectrics. They have

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E} \tag{3.11}$$

where χ_e is called the electric susceptibility. It is always positive: $\chi_e > 0$

Any function that has $\mathbf{P}(\mathbf{E} = 0) = 0$ can be Taylor expanded as a linear term + quadratic + cubic and so on. For suitably small electric fields, the linear term always dominates. To determine when the quadratic and higher order terms become important, we need to know the relevant scale in the problem. For us, this is the scale of electric fields inside the atom. In most situations, the applied electric field leading to the polarisation is a tiny perturbation and the linear term dominates. The linearity fails for suitably high electric fields.

There are exceptions to linear dielectrics. Perhaps the most striking exception are materials for which $\mathbf{P} \neq 0$ even in the absence of an electric field. Such materials − which are not particularly common - are called ferroelectric. For what it's worth, an example is $BaTiO_3$.

Well, for linear dielectrics

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 \left(1 + \chi_e\right) \mathbf{E} \tag{3.12}$$

So we define

$$\mathbf{D} = \epsilon \mathbf{E} \tag{3.13}$$

where

$$\epsilon \equiv \epsilon_0 \left(1 + \chi_e\right) \tag{3.14}$$

This constant ϵ is called the permittivity of the material. (In vacuum, where there is no matter to polarize, the susceptibility is zero, and the permittivity is ϵ_0 . That's why ϵ_0 is called the permittivity of free space.

The relative permittivity, or dielectric constant, of the material, is defined as

$$\epsilon_r \equiv 1 + \chi_e = \frac{\epsilon}{\epsilon_0} \tag{3.15}$$

Example: A metal sphere of radius R_1 carries a charge Q. It is surrounded, out to radius R_2 by linear dielectric material of permittivity ϵ. Find the potential at the center (relative to infinity).

Solution: To compute V, we need to know $\mathbf{E}$; to find $\mathbf{E}$, we might first try to locate the bound charge; we could get the bound charge from $\mathbf{P}$, but we can't calculate $\mathbf{P}$ unless we already know $\mathbf{E}$ (from the relation $\mathbf{P} = \epsilon_0\chi_e\mathbf{E}$)

We can calculate $\mathbf{D}$, as $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}}$

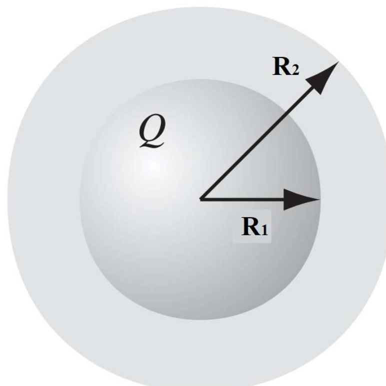


Figure 3.2: Conductor surrounded by dielectrics

$$\mathbf{D} = \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, \qquad \text{for all points } r > R_1$$

(Inside the metal sphere, of course, $\mathbf{E} = \mathbf{P} = \mathbf{D} = \mathbf{D} = \mathbf{0}$. Once we know $\mathbf{D}$, it is a trivial matter to obtain $\mathbf{E}$, using $\mathbf{D} = \epsilon\mathbf{E}$

$$\mathbf{E} = \begin{cases} \dfrac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & \text{for } R_1 < r < R_2 \\ \dfrac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & \text{for } r > R_2 \end{cases}$$

The potential is then

$$\begin{aligned} V &= -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^{R_2}\left(\frac{Q}{4\pi\epsilon_0 r^2}\right)dr - \int_{R_2}^{R_1}\left(\frac{Q}{4\pi\epsilon r^2}\right)dr - \int_{R_1}^0 (0)dr \\ &= \frac{Q}{4\pi}\left(\frac{1}{\epsilon_0 R_2} + \frac{1}{\epsilon R_1} - \frac{1}{\epsilon R_2}\right) \end{aligned}$$

The polarization is present inside the dielectric material

$$\mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \frac{\epsilon_0\chi_e Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}; \qquad R_1 < r < R_2$$

The bound volume charge

$$\rho_b = -\nabla\cdot\mathbf{P} = 0$$

The volume surface charge

$$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} \dfrac{\epsilon_0\chi_e Q}{4\pi\epsilon R_2^2}, & \text{at the outer surface,} \\ \dfrac{-\epsilon_0\chi_e Q}{4\pi\epsilon R_1^2}, & \text{at the inner surface.} \end{cases}$$

3.4 Boundary Conditions in dielectric

$$D^{\perp}_{\text{above}} - D^{\perp}_{\text{below}} = \sigma_f \tag{3.16}$$

we write the same equation in terms of $\vec{E}$

$$\epsilon_{\text{above}}E^{\perp}_{\text{above}} - \epsilon_{\text{below}}E^{\perp}_{\text{below}} = \sigma_f \tag{3.17}$$

The boundary conditions in terms of Electric field...

$$E^{\perp}_{\text{above}} - E^{\perp}_{\text{below}} = \frac{1}{\epsilon_0}\sigma \tag{3.18}$$

$$\mathbf{E}^{\parallel}_{\text{above}} - \mathbf{E}^{\parallel}_{\text{below}} = \mathbf{0} \tag{3.19}$$

Boundary conditions in terms of potentials

$$\epsilon_{\text{ above}}\frac{\partial V_{\text{ above}}}{\partial n} - \epsilon_{\text{ below}}\frac{\partial V_{\text{ below}}}{\partial n} = -\sigma_f \tag{3.20}$$

$$V_{\text{above}} = V_{\text{below}} \tag{3.21}$$

We use the boundary conditions depending on the problems given.

3.5 Energy in Dielectrics

Energy stored in a dielectric system is

$$W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}d\tau \tag{3.22}$$

Example: A sphere of radius R is filled with material of dielectric constant ϵ_r and uniform embedded free charge ρ_f. What is the energy of this configuration?

Solution: The displacement vector

$$\mathbf{D}(r) = \begin{cases} \dfrac{\rho_f}{3}\mathbf{r} & (r < R) \\ \dfrac{\rho_f}{3}\frac{R^3}{r^2}\hat{\mathbf{r}} & (r > R) \end{cases}$$

The electric field

$$\mathbf{E}(r) = \begin{cases} \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < R) \\ \frac{\rho_f}{3\epsilon_0}\frac{R^3}{r^2}\hat{\mathbf{r}} & (r > R) \end{cases}$$

Then the electrostatic energy without considering the dielectrics ($W = \frac{\epsilon_0}{2}\int E^2 d\tau$)

$$\begin{aligned} W_1 &= \frac{\epsilon_0}{2}\left[\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)^2 \int_0^R r^2 4\pi r^2 dr + \left(\frac{\rho_f}{3\epsilon_0}\right)^2 R^6 \int_R^\infty \frac{1}{r^4} 4\pi r^2 dr\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5 \left(\frac{1}{5\epsilon_r^2} + 1\right) \end{aligned}$$

The total energy ($W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E} d\tau$)

$$\begin{aligned} W_2 &= \frac{1}{2}\left[\left(\frac{\rho_f}{3}\right)\left(\frac{\rho_f}{3\epsilon_0\epsilon_r}\right)\int_0^R r^2 4\pi r^2 dr + \left(\frac{\rho_f R^3}{3}\right)\left(\frac{\rho_f R^3}{3\epsilon_0}\right)\int_R^\infty \frac{1}{r^4}4\pi r^2 dr\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5 \left(\frac{1}{5\epsilon_r} + 1\right) \end{aligned}$$

Let's check that W_2 is the work done on the free charge in assembling the system. We start with the (uncharged, unpolarized) dielectric sphere, and bring in the free charge in infinitesimal installments (dq) , filling out the sphere layer by layer. When we have reached radius r', the electric field is

$$\mathbf{E}(r) = \begin{cases} \frac{\rho_f}{3\epsilon_0\epsilon_r}\mathbf{r} & (r < r') \\ \frac{\rho_f}{3\epsilon_0\epsilon_r}\frac{r'^3}{r^2}\hat{\mathbf{r}} & (r' < r < R) \\ \frac{\rho_f}{3\epsilon_0}\frac{r'^3}{r^2}\hat{\mathbf{r}} & (r > R) \end{cases}$$

The work required to bring the next dq in from infinity to r' is

$$\begin{aligned} dW &= -dq\left[\int_\infty^R \mathbf{E}\cdot d\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot d\mathbf{l}\right] \\ &= -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'}\frac{1}{r^2}dr\right] \\ &= \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right] dq \end{aligned}$$

$$\begin{aligned} dW &= -dq\left[\int_\infty^R \mathbf{E}\cdot d\mathbf{l} + \int_R^{r'} \mathbf{E}\cdot d\mathbf{l}\right] \\ &= -dq\left[\frac{\rho_f r'^3}{3\epsilon_0}\int_\infty^R \frac{1}{r^2}dr + \frac{\rho_f r'^3}{3\epsilon_0\epsilon_r}\int_R^{r'}\frac{1}{r^2}dr\right] \\ &= \frac{\rho_f r'^3}{3\epsilon_0}\left[\frac{1}{R} + \frac{1}{\epsilon_r}\left(\frac{1}{r'} - \frac{1}{R}\right)\right] dq \end{aligned}$$

This increases the radius (r') :

$$dq = \rho_f 4\pi r'^2 dr'$$

so the total work done, in going from $r' = 0$ to $r' = R$, is

$$\begin{aligned} W &= \frac{4\pi\rho_f^2}{3\epsilon_0}\left[\frac{1}{R}\left(1-\frac{1}{\epsilon_r}\right)\int_0^R r'^5 dr' + \frac{1}{\epsilon_r}\int_0^R r^4 dr'\right] \\ &= \frac{2\pi}{9\epsilon_0}\rho_f^2 R^5\left(\frac{1}{5\epsilon_r}+1\right) = W_2 \end{aligned}$$

Example: A spherical conductor, of radius a, carries a charge Q. It is surrounded by linear dielectric material of susceptibility χ_e, out to radius b. Find the energy of this configuration.

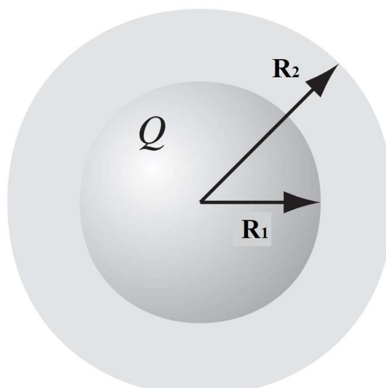


Figure 3.3: Conductor surrounded by dielectrics: Energy

Solution: The displacement vector is

$$\mathbf{D} = \left\{\begin{array}{ll} 0, & (r<a) \\ \frac{Q}{4\pi r^2}\hat{\mathbf{r}}, & (r>a) \end{array}\right\}$$

So the electric field

$$\mathbf{E} = \left\{\begin{array}{ll} 0, & (r<a) \\ \frac{Q}{4\pi\epsilon r^2}\hat{\mathbf{r}}, & (a<r<b) \\ \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}, & (r>b) \end{array}\right\}$$

The work done

$$\begin{aligned} W = \frac{1}{2}\int \mathbf{D}\cdot\mathbf{E}d\tau &= \frac{1}{2}\frac{Q^2}{(4\pi)^2}4\pi\left\{\frac{1}{\epsilon}\int_a^b \frac{1}{r^2}\frac{1}{r^2}r^2 dr + \frac{1}{\epsilon_0}\int_b^\infty \frac{1}{r^2}dr\right\} \\ &= \frac{Q^2}{8\pi}\frac{1}{\epsilon}\left(\frac{-1}{r}\right)\Big|_a^b + \frac{1}{\epsilon_0}\left(\frac{-1}{r}\right)\Big|_b^\infty \\ &= \frac{Q^2}{8\pi\epsilon_0}\left\{\frac{1}{(1+\chi_e)}\left(\frac{1}{a}-\frac{1}{b}\right)+\frac{1}{b}\right\} \\ &= \frac{Q^2}{8\pi\epsilon_0(1+\chi_e)}\left(\frac{1}{a}+\frac{\chi_e}{b}\right) \end{aligned}$$

3.5.1 Forces on Dielectrics

Force is the derivative of energy

$$F = \frac{dW}{dx} \tag{3.23}$$

If capacitance of a system is changing with respect to x then the force

$$F = \frac{1}{2}V^2\frac{dC}{dx} \tag{3.24}$$

Example: Capacitor partially filled with dielectrics

Figure below shows parallel plates of size $\ell \times w$ and separation d , partially filled with a dielectric slab. The dimensions of the slab are $x \times w \times d$, with $x < \ell$. What is the capacitance? Assume the separation d is small compared to x and $\ell - x$, so that end effects are negligible.

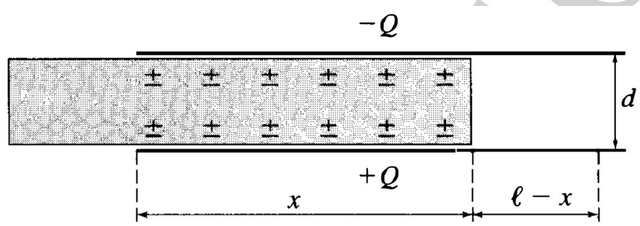


Figure 3.4: A dielectric slab inserted part-way between charged plates. A force is pulling the slab into the space between the plates

Let the potential difference between the plates be V . The key to this example is that the electric field $\mathbf{E}$ is the same in the two regions (dielectric and vacumm) between the plates. By planar symmetry, the field is uniform in both regions and normal to the plates. But E, being normal to the plates, is tangent to the boundary surface of the dielectric. Since the tangential field must be continuous, the field is the same on either side of the boundary. Thus the electric field is $\mathbf{E} = \hat{\mathbf{n}}V/d$ in both regions, where $\hat{\mathbf{n}}$ is normal to the plates.

To determine the capacitance we must relate V to the charge $\pm Q$ on the plates. Apply Gauss's Law to a surface surrounding one plate, e.g. the lower plate: The flux of $\mathbf{D}$ is equal to Q. That is, since $\mathbf{D}$ is $\epsilon\mathbf{E}$ in the dielectric (of area xw) and $\epsilon_0\mathbf{E}$ in the vacuum region [of area $(\ell - x)w$]

$$\epsilon Exw + \epsilon_0 E(\ell - x)w = Q$$

Substituting $E = V/d$ we find that the capacitance is

$$C = \frac{Q}{V} = [\epsilon x + \epsilon_0(\ell - x)]\,\frac{w}{d}$$

We might have anticipated this result by noting that the system is equivalent to two capacitors in parallel, with $C_1 = \epsilon xw/d$ and $C_2 = \epsilon_0(\ell - x)w/d$. Then the combined capacitance is $C_1 + C_2$.

Now what is the force on the dielectric slab? We can have two cases.

The plates are with fixed charge: Plates are isolated so that their charges $\pm Q$ are constant. The work done by the electrostatic force if the slab moves a distance dx into the capacitor is $F dx$ Because the system is isolated, conservation of energy says that the work is equal to $-dU$ where U is the total energy of the capacitor. Therefore the inward force on the slab is $F = -dU/dx$

To calculate U for this example it is natural to use the equation $U = Q^2/2C$ because Q is constant. The capacitance we calculated. Thus the energy is

$$U(x) = \frac{Q^2 d}{2w\left[\epsilon x + \epsilon_0(\ell - x)\right]}$$

So the force on the slab is

$$F = \frac{Q^2 d\,(\epsilon - \epsilon_0)}{2w\left[\epsilon x + \epsilon_0(\ell - x)\right]^2}$$

For fixed charge the force decreases as x increases. We may also express F in terms of the potential difference between the plates, $V = Q/C$, as

$$F = \frac{V^2 w\,(\epsilon - \epsilon_0)}{2d}$$

Plates with fixed potential difference: Now suppose the plates in Fig. 3.4 are connected to a battery so that the potential difference V is fixed. In this case the work done by the electrostatic force F is not equal to $-dU$, because additional energy is supplied by the battery. If the slab moves by distance dx, then charge dQ is transferred to the plates from the battery, and so the battery supplies energy $(d)V$. The conservation of energy in this case is

$$dU = (dQ)V - F dx$$

Here it is natural to use the equation $U = CV^2/2$, because V is constant. Also, $Q = CV$ so $dQ = (dC)V$. Inserting these relations into the conservation law gives

$$F dx = -\frac{1}{2}(dC)V^2 + (dC)V^2 = \frac{1}{2}(dC)V^2$$

Thus the force on the slab is

$$F = \frac{1}{2}\frac{dC}{dx}V^2 = \frac{V^2 w\,(\epsilon - \epsilon_0)}{2d}$$

Again, a reminder, do all the exercises which are given inside the chapters of Griffiths.